Make Circle

John Finn

April 22, 2004

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1 Opening Comments

This gives the programming answer to a question posed by Robert Heller:

- Given the radius and two points on a circle, can we determine the circle?

A moment's reflection reveals that to determine the circle here means to find the center.

The answer is "almost", which is not hard to see if we begin by sketching the desired circle, calling its center M. I think it's best to make the circle a dotted line, since we're going to pretend we don't know just where it is. We sketch the points A and C somewhere along its circumference, making them boldface, say, to indicate that we do know just where they are. The center M we make faint, since we start out not knowing where it is. Since do know the radius r, we begin by setting a compass to that size.

Now we use the compass to draw two r-sized circles, whose centers are A and C. If you take the time to actually make the sketch, or if you have a good visual imagination, you'll see that these two circles intersect in two points, one of which is M —well, that is, unless A and C happen to be exactly the endpoints of a diameter, in which case the two circles intersect at just the single point M.

But in the generic case, if A and C are just any old points on the unknown circle, then the unknown center M is one of the two points of intersection of the two circles we construct. This is why the answer is “almost”; without further information, we have two possible answers. For the further information, Heller suggests that we stipulate that C lie counterclockwise from A on the circle, meaning that in drawing the shortest arc from A to C our pencil moves counterclockwise. (This is again ambiguous in the case that A and C are diametrically opposite, but again in that special case we only have a single intersection, and so don’t need to know the relative positions of A and C).

Notice also that while we *could*, for given points A and C, give values for r for which no solution exists, it’s easy to recognize these impossible cases; they’re exactly the ones where r is less than half the distance between A and C.

This program lets the user use the mouse to

- pick points A and C, and then
- specify a value for r by picking a point whose distance from A will be taken as r.

The program will then draw the two intersecting circles with centers at A and C, figure out algebraically their two points of intersection, and pick as M the one that gives A and C the right orientation. It will then show which is C, and plot the circle we’re trying to find.

An explanation of the algebra —which amounts to finding the two solutions to two quadratic equations in two unknowns— is appended at the bottom of the program itself; these opening remarks are already long enough!

2 Detailed Explaination

2.1 Problem

Suppose we have points $\vec{a} = (a, b)$ and $\vec{c} = (c, d)$ in the plane, and know they lie on a circle whose radius r we also know. Find the circle.

2.2 Solution

First, since we know r, to determine the circle it is enough to find its center, $\vec{m} = (m, n)$. To do so, construct a circle of radius r about each of $\vec{a}$ and $\vec{c}$:

Now the two (generically) points of intersection of these two circles are *both* centers of circles radius r that contain $\vec{a}$ and $\vec{c}$. For a unique solution, we need to add a condition to those given, so we specify that the orientation of $\vec{a}$ and $\vec{c}$ by requiring that $\vec{c}$ be counterclockwise from $\vec{a}$ on the circle–meaning that the *shorter* of the 2 arcs joining $\vec{a}$ to $\vec{c}$ is in the counterclockwise (or mathematically positive) direction.

$\overset{\frown}{\vec{a}\vec{c}}$ counterclockwise $\qquad$ $\overset{\frown}{\vec{a}\vec{c}}$ clockwise

To determine $\vec{m}$, think of it for the nonce as the unknown $\vec{x} = (x, y)$. We then have 2 quadratic equations in x and y:

$$(x - a)^2 + (y - b)^2 = r^2 \tag{1}$$

and

$$(x-c)^2+(y-d)^2=r^2 \tag{2}$$

expanding these out, and substracting (1) from (2) we get (after gathering terms and factoring):

$$\begin{array}{c} x^2-2cx+c^2+y^2-2dy+d^2=r^2 \\ -(x^2-2ax+a^2+y^2-2by+b^2=r^2) \\ \hline 2(a-c)x+2(b-d)y=(a^2+b^2)-(c^2+d^2) \end{array} \tag{3}$$

Now this may look complicated, but it's just

$$Jx+Gy=T \tag{4}$$

and (4) is linear in x and y, so trivial to solved for y in terms of x:

$$y=\frac{T-Jx}{G} \tag{5}$$

Doing this eliminates y (at least for the nonce) and gives a quadratic equation in just the single variable x. That is, if we plug in $\frac{T-Jx}{G}$ for y, in either (1) or (2), we get a quadratic in x–which we can solve by plugging in its coefficients into the quadratic formula. The quadratic will give *2* values for x, one for each of the two points m_1 and m_2 shown earlier.

So we take (1) plug in $y=\frac{T-Jx}{G}$:

$$(x-a)^2+\left(\frac{T-Jx}{G}-b\right)^2=r^2 \tag{6}$$

and now expand this out enough to identify x^2, x, and constant terms

$$x^2-2ax+a^2+\left(\frac{T-Jx}{G}\right)^2-2\left(\frac{T-Jx}{G}\right)b+b^2=r^2 \tag{7}$$

$$x^2-2ax+a^2+\frac{T^2-2JTx+J^2x^2}{G^2}-2\left(\frac{T-Jx}{G}\right)b+b^2=r^2 \tag{8}$$

so we have

$$\left(1+\frac{J^2}{G^2}\right)x^2+\left(-2a-\frac{2JT}{G^2}+\frac{2bJ}{G}\right)x+\left((a^2+b^2)+\frac{T^2}{G^2}-\frac{2bJ}{G}-r^2\right)=0 \tag{9}$$

$$ux^2 \qquad +vx \qquad +w=0 \tag{10}$$

Now we *could* express each of these coeff's in terms of a, b, c, d, and grind away to try to simplfy them, but if we just want them for a computer program, there's no need to–which is good, because they *don't* simplify a great deal.

Our two answers, then are

$$x=\frac{-v\pm\sqrt{v^2-4uw}}{2u}, \tag{11}$$

and

$$y=\frac{T-Jx}{G} \tag{12}$$

with each of the x's plugged in.

Finally, to pick the right (x, y) for (m, n)–that is, to deal with the orientation, we note that the $\vec{c}$ counterclockwise from $\vec{a}$ means exactly that the angle from $\vec{a}$ to $\vec{c}$ be between 0 and 180°–well, modulo 2π. But this is also true if the and only if the sine of this angle is positive. So we see which of

$$\sin(\arctan(\vec{a}-\vec{m}_1, \vec{c}-\vec{m}_1)) \tag{13}$$

and

$$\sin(\arctan(\vec{a}-\vec{m}_2, \vec{c}-\vec{m}_2)) \tag{14}$$

is positive, and we're done.